

# QUANTUM QUASI-SHUFFLE ALGEBRAS II. EXPLICIT FORMULAS, DUALIZATION, AND REPRESENTATIONS

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**ABSTRACT.** Using the concept of mixable shuffles, we formulate explicitly the quantum quasi-shuffle product, as well as the subalgebra generated by primitive elements of the quantum quasi-shuffle bialgebra. We construct a braided coalgebra structure which is dual to the quantum quasi-shuffle algebra. We provide representations of quantum quasi-shuffle algebras on commutative braided Rota-Baxter algebras. As an application, we establish a formal power series whose terms come from a special representation of some kind of quasi-shuffle algebra and whose evaluation at 1 is the multiple  $q$ -zeta values.

## 1. INTRODUCTION

In [24], Ree introduced the shuffle algebra which has been studied extensively during the past fifty years. The shuffle product is carried out on the tensor space  $T(V)$  of a vector space  $V$  by using the shuffle rule. Its natural generalization is the quasi-shuffle product where  $V$  is moreover an associative algebra and the new product on  $T(V)$  involves both of the shuffle product and the multiplication of  $V$ . Quasi-shuffle algebras first arose in the work of Newman and Radford [22] for the study of cofree irreducible Hopf algebras built on associative algebras, where they were constructed by the universal property of cofree pointed irreducible coalgebras. Later, they were rediscovered independently by other mathematicians with various motivations. In 2000, motivated by his work on multiple zeta values, Hoffman defined the quasi-shuffle algebra by an inductive formula ([10]). In the same year, Guo and Keigher introduced the mixable shuffle algebra by using an explicit formula in their study of Rota-Baxter algebras ([6] and [8]). After these seminal works, the research of quasi-shuffle algebras become active. Besides their own interest, quasi-shuffle algebras have many significant applications in other branches of mathematics, such as multiple zeta values ([11]), Rota-Baxter algebras ([6] and [3]), and commutative tridendriform algebras ([18]). They also appear in the study of shuffle identities between Feynman graphs ([17]).

For both of physical and mathematical considerations, people want to deform or quantize some important algebra structures. The most famous example is absolutely the quantum group. To our surprise, there is an implicit but significant connection between quantum groups and shuffle algebras. Rosso [25] constructed the quantization of shuffle algebras. This is a new kind of quantized algebras and leads to an intrinsic understanding of the quantum group. Since shuffle algebras

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are special quasi-shuffle algebras, and the importance of the later one, people would expect to find out what the quantization of quasi-shuffle algebras is and whether it can bring us some useful information. Some  $q$ -analogues of the quasi-shuffle product were discussed in [28], [10] and [2]. But these formulas are not systematic. For instance, as Hoffman said in [10], his  $q$ -deformation of quasi-shuffles is more or less experiential. The general construction of quantized quasi-shuffles is due to Rosso ([26]) in the spirit of his quantum shuffles. We describe Rosso's idea as follows. Let  $M$  be a Hopf bimodule over a Hopf algebra  $H$ . In addition, if  $M$  is an algebra and the multiplication is compatible with the module and comodule structure of  $M$  in some sense, then one can construct a new algebra structure on the cotensor coalgebra  $T_H^c(M)$  by using its universal property. Let  $M^R$  be the space of right coinvariants of  $M$ . Then the subspace  $T(M^R)$  equipped with this new multiplication is a generalization of the classical quasi-shuffle product. In general, given a braided algebra  $(A, m, \sigma)$ , one can construct an analogue of the quasi-shuffle algebra on  $T(A)$ , where the action of the usual flip is replaced by that of the braiding. The resulting algebra is called a quantum quasi-shuffle algebra. In particular, the  $q$ -analogues mentioned above are some sort of special cases of Rosso's quantum quasi-shuffle. In [13], the construction of quantum quasi-shuffles appears, as a special braided cofree Hopf algebra, in the framework of quantum multi-brace algebras.

Some interesting properties of quantum quasi-shuffle algebras have been studied in [14], including the commutativity, universal property, and etc. In a recent paper [12], applications of quantum quasi-shuffle algebras to Rota-Baxter algebras and tridendriform algebras are found. This paper continues the trip. We first establish some explicit results concerning this new subject. We start by reformulating the product. Originally, the quantum quasi-shuffle algebra is constructed by using the universal property of connected coalgebras ([13]). Later, it is defined through an inductive formula ([14]). But neither of these two approaches can provide an explicit formula. To know more about this new subject, a more clear form of the multiplication formula is definitely helpful. Here, we use the notion of mixable shuffles introduced in [8] to establish a complete description of the quantum quasi-shuffle product. In the case of quantum shuffles, the subalgebra generated by primitive elements is especially important. Under some suitable assumptions, it is isomorphic, as a Hopf algebra, to the positive part of quantum groups ([25]). It seems quite reasonable that the corresponding subalgebra in a quantum quasi-shuffle algebra will have some desirable properties. Recently, Fang and Rosso ([5]) use it to realize the whole quantum group associated to a symmetrizable Kac-Moody Lie algebra. In this paper, we use mixable shuffles to describe such subalgebras. Sometimes, dual constructions of algebraic objects bring people extra information different from the original ones. On the other hand, the universal property of connected coalgebras is not so familiar by non-Hopf algebraists. So we use the universal property of tensor algebras to construct a braided coalgebra structure on  $T(C)$  for a braided coalgebra  $C$ , and show that its dual is the quantum quasi-shuffle algebra. This enables one to study the quantum quasi-shuffle algebra through its dual. We would like to mention that Manchon had studied such a structure when the braiding is the usual flip map ([21]). Representation theory is absolutely an essential tool for the investigation of algebras. Inspired by the work of Guo and Keigher, we construct representations of quantum quasi-shuffle algebras on some sort of braided Rota-Baxter algebras which are introduced in [12]. As an application, we use some special kind of such

representations to construct a formal power series whose terms are images of an algebra map from a quasi-shuffle algebra to a Rota-Baxter algebra. The evaluation of this formal power series at 1 is the multiple  $q$ -zeta values.

This paper is organized as follows. In Section 2, several concrete examples of braided algebras are provided. In Section 3, we recall the construction of quantum quasi-shuffle algebras. After that, we use the notion of mixable shuffles to establish an explicit formula for the quantum quasi-shuffle in Section 4 and describe the subalgebra generated by primitive elements in Section 5. In Section 6, we construct the dual coalgebra of a quantum quasi-shuffle algebra. In Section 7, we construct algebra maps from quantum quasi-shuffle algebras to commutative braided Rota-Baxter algebras which provide representations of the former ones. We use this construction to establish a formal power series whose evaluation at 1 is the multiple  $q$ -zeta values.

**Notation.** In this paper, we denote by  $\mathbb{K}$  a ground field of characteristic 0. All the objects we discuss are defined over  $\mathbb{K}$ . For a vector space  $V$ , we denote by  $T(V)$  the tensor algebra of  $V$ , by  $\otimes$  the tensor product within  $T(V)$ , and by  $\underline{\otimes}$  the one between  $T(V)$  and  $T(V)$ .

We denote by  $\mathfrak{S}_n$  the symmetric group acting on the set  $\{1, 2, \dots, n\}$  and by  $s_i$ ,  $1 \leq i \leq n-1$ , the standard generators of  $\mathfrak{S}_n$  permuting  $i$  and  $i+1$ . For fixed  $k, n \in \mathbb{N}$ , we define the shift map  $\text{shift}_k : \mathfrak{S}_n \rightarrow \mathfrak{S}_{n+k}$  by  $\text{shift}_k(s_i) = s_{i+k}$  for any  $1 \leq i \leq n-1$ . For the reason of intuition and the simplicity of notation, we denote  $1_{\mathfrak{S}_k} \times w = \text{shift}_k(w)$  for any  $w \in \mathfrak{S}_n$ . The notations  $w \times 1_{\mathfrak{S}_k}$ ,  $1_{\mathfrak{S}_k} \times w \times 1_{\mathfrak{S}_l}$  and others are understood similarly.

A braiding  $\sigma$  on a vector space  $V$  is an invertible linear map in  $\text{End}(V \otimes V)$  satisfying the quantum Yang-Baxter equation on  $V^{\otimes 3}$ :

$$(\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma)(\sigma \otimes \text{id}_V) = (\text{id}_V \otimes \sigma)(\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma).$$

A braided vector space  $(V, \sigma)$  is a vector space  $V$  equipped with a braiding  $\sigma$ . For any  $n \in \mathbb{N}$  and  $1 \leq i \leq n-1$ , we denote by  $\sigma_i$  the operator  $\text{id}_V^{\otimes i-1} \otimes \sigma \otimes \text{id}_V^{\otimes n-i-1} \in \text{End}(V^{\otimes n})$ . For any  $w \in \mathfrak{S}_n$ , we denote by  $T_w^\sigma$  the corresponding lift of  $w$  in the braid group  $B_n$ , defined as follows: if  $w = s_{i_1} \cdots s_{i_l}$  is any reduced expression of  $w$ , then  $T_w^\sigma = \sigma_{i_1} \cdots \sigma_{i_l}$ . This definition is well-defined (see, e.g., Theorem 4.12 in [16]).

We define  $\beta : T(V) \underline{\otimes} T(V) \rightarrow T(V) \underline{\otimes} T(V)$  by requiring that the restriction of  $\beta$  on  $V^{\otimes i} \underline{\otimes} V^{\otimes j}$ , denoted by  $\beta_{ij}$ , is  $T_{\chi_{ij}}^\sigma$ , where

$$\chi_{ij} = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & i+2 & \cdots & i+j \\ j+1 & j+2 & \cdots & j+i & 1 & 2 & \cdots & j \end{pmatrix} \in \mathfrak{S}_{i+j},$$

for any  $i, j \geq 1$ . For convenience, we denote by  $\beta_{0i}$  and  $\beta_{i0}$  the identity map of  $V^{\otimes i}$ .

Given an invertible element  $q \in \mathbb{K}$  which is not a root of unity, we denote  $[0]_q = 1$  and  $[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$  when  $n \in \mathbb{N}$ . We also denote  $[n]_q! = [1]_q \cdots [n]_q$ .

## 2. BRAIDED ALGEBRAS

We start by recalling the notion of braided algebras which is the relevant object of associative algebras in braided categories. In the following, all algebras are always assumed to be associative, but not necessarily unital.

**Definition 2.1.** Let  $A = (A, m)$  be an algebra with product  $m$ , and  $\sigma$  be a braiding on  $A$ . We call the triple  $(A, m, \sigma)$  a *braided algebra* if it satisfies the following conditions:

$$\begin{aligned} (\text{id}_A \otimes m)\sigma_1\sigma_2 &= \sigma(m \otimes \text{id}_A), \\ (m \otimes \text{id}_A)\sigma_2\sigma_1 &= \sigma(\text{id}_A \otimes m). \end{aligned}$$

Moreover, if  $A$  is unital and its unit  $1_A$  satisfies that for any  $a \in A$ ,

$$\begin{aligned} \sigma(a \otimes 1_A) &= 1_A \otimes a, \\ \sigma(1_A \otimes a) &= a \otimes 1_A, \end{aligned}$$

then  $A$  is called a *unital braided algebra*.

Because all the constructions in this paper are based on braided algebras, we provide several concrete examples which will either be used in our later discussion or afford the reader some illustrations. Some of them may be known, while some may be new. For more examples, one can see [2] and [13].

**Example 2.2.** Let  $V$  be a vector space with basis  $\{e_i\}$  which is at most countable. We provide a braided algebra structure on  $V$ . The braiding  $\sigma$  on  $V$  is given by  $\sigma(e_i \otimes e_j) = q_{ij}e_j \otimes e_i$ , where  $q_{ij}$ 's are nonzero scalars in  $\mathbb{K}$  such that  $q_{ij}q_{ik} = q_{i+j+k}$  and  $q_{ik}q_{jk} = q_{i+j-k}$  for any  $i, j, k$ . For instance, let  $q$  be a nonzero scalar in  $\mathbb{K}$  and  $q_{ij} = q^{ij}$ . The multiplication  $\cdot$  on  $V$  which is compatible with the braiding  $\sigma$  is given as follows.

Case 1. If  $V$  is a finite-dimensional vector space with basis  $\{e_1, e_2, \dots, e_N\}$ , then we define

$$e_i \cdot e_j = \begin{cases} e_{i+j}, & \text{if } i+j \leq N, \\ 0, & \text{otherwise.} \end{cases}$$

Case 2. If  $V$  is a vector space with basis  $\{e_i\}_{i \in \mathbb{N}}$ , then we define  $e_i \cdot e_j = e_{i+j}$  for any  $i, j \in \mathbb{N}$ .

It is evident that  $\cdot$  is an associative algebra structure on  $V$  in both cases. Notice that

$$\begin{aligned} (\text{id}_V \otimes \cdot)\sigma_1\sigma_2(e_i \otimes e_j \otimes e_k) &= q_{jk}q_{ik}e_k \otimes e_{i+j} \\ &= q_{i+j-k}e_k \otimes e_{i+j} \\ &= \sigma(\cdot \otimes \text{id}_V)(e_i \otimes e_j \otimes e_k), \end{aligned}$$

and similarly  $(\cdot \otimes \text{id}_V)\sigma_2\sigma_1 = \sigma(\text{id}_V \otimes \cdot)$ . Therefore  $(V, \cdot, \sigma)$  is a braided algebra.

In particular, the polynomial algebra  $\mathbb{K}[t]$  is a braided algebra with respect to the braiding defined by  $\sigma(t^n \otimes t^m) = q^{nm}t^m \otimes t^n$ .

**Example 2.3.** All notions of this example can be found in [15]. Let  $q \neq 1$  be an invertible scalar in  $\mathbb{K}$ , and  $x, y$  be two indeterminates. Denote by  $\mathbb{K}_q[x, y]$  the quantum plane, i.e., the algebra generated by  $x, y$  with the relation  $yx = qxy$ . It has a linear basis  $\{x^i y^j\}_{i,j \geq 0}$ . Define two algebra automorphisms  $\omega_x$  and  $\omega_y$  of  $\mathbb{K}_q[x, y]$  by requiring that

$$\omega_x(x) = qx, \omega_x(y) = y, \omega_y(x) = x, \omega_y(y) = qy,$$

and define two endomorphisms  $\partial_q/\partial x$  and  $\partial_q/\partial y$  by requiring that

$$\frac{\partial_q(x^m y^n)}{\partial x} = [m]x^{m-1}y^n, \frac{\partial_q(x^m y^n)}{\partial y} = [n]x^m y^{n-1},$$

where  $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$  for any  $k \in \mathbb{N}$ .

Let  $U_q \mathfrak{sl}_2$  be the quantized algebra associated to  $\mathfrak{sl}_2$ , i.e., the algebra generated by  $E, F, K, K^{-1}$  with the relations

$$KK^{-1} = K^{-1}K = 1,$$

$$KE = q^2 EK, KF = q^{-2} FK,$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

It is well-known that  $U_q \mathfrak{sl}_2$  is a quasi-triangular Hopf algebra.

By Theorem VII 3.3 in [15],  $\mathbb{K}_q[x, y]$  is a  $U_q \mathfrak{sl}_2$ -module-algebra with the following module structure: for any  $P \in \mathbb{K}_q[x, y]$ ,

$$EP = x \frac{\partial_q(P)}{\partial y}, FP = \frac{\partial_q(P)}{\partial x} y,$$

$$KP = (\omega_x \omega_y^{-1})(P), K^{-1}P = (\omega_y \omega_x^{-1})(P).$$

Set  $V = \text{Span}_{\mathbb{K}}\{x, y\}$ . It is not hard to see that the above action restricting on  $V$  is the standard 2-dimensional simple  $U_q \mathfrak{sl}_2$ -module structure. We know that (Theorem 2.7 in [13]) every module-algebra over a quasi-triangular Hopf algebra has a braided algebra structure. So  $\mathbb{K}_q[x, y]$  is a braided algebra.

**Example 2.4.** Let  $(V, \sigma)$  be a braided vector space. It is known that  $(T(V), m, \beta)$  is a braided algebra, where  $m$  is the concatenation product. Let  $M_n : V^{\otimes n} \rightarrow T(V)$  be a linear map such that  $\beta(M_n \otimes \text{id}_V) = (\text{id}_V \otimes M_n)\beta_{n1}$  and  $\beta(\text{id}_V \otimes M_n) = (M_n \otimes \text{id}_V)\beta_{1n}$ . If we denote by  $\mathcal{I}$  the ideal of  $T(V)$  generated by  $\text{Im} M_n$ , the image of  $M_n$ , then  $\beta(T(V) \underline{\otimes} \mathcal{I} + \mathcal{I} \underline{\otimes} T(V)) \subset T(V) \underline{\otimes} \mathcal{I} + \mathcal{I} \underline{\otimes} T(V)$ . So the quotient algebra  $T(V)/\mathcal{I}$  is also a braided algebra. For instance, if  $M_2 = \text{id}_V^{\otimes 2} - \sigma$ , then the quotient algebra is the  $r$ -symmetric algebra defined in [2].

**Example 2.5.** Let  $H$  be a finite dimensional quasi-triangular Hopf algebra. By a result of Majid (Theorem 3.3 in [20]), the quantum double  $\mathcal{D}(H)$  of  $H$  is a braided algebra (according to a discussion in [13] for Radford's work [23]).

## 3. QUANTUM QUASI-SHUFFLE ALGEBRAS

For any algebra  $A$ , it is a braided algebra with respect to the flip map switching the two factors of  $A \otimes A$ . One can construct an algebra structure on  $T(A)$  which combines the multiplication of  $A$  and the shuffle product of  $T(A)$  (see [22]). This structure is the so-called quasi-shuffle algebra. If the flip map is replaced by a general braiding, one can construct a quantized quasi-shuffle product by assuming some compatibilities between the multiplication of  $A$  and the braiding (for more details, one can see [13] and [14]). More precisely, given a braided algebra  $(A, m, \sigma)$ , the *quantum quasi-shuffle product*  $\bowtie_\sigma$  on  $T(A)$  is given by the following formulas.

For any  $\lambda \in \mathbb{K}$  and  $x \in T(A)$ ,

$$\lambda \bowtie_\sigma x = x \bowtie_\sigma \lambda = \lambda \cdot x.$$

For  $i, j \geq 2$  and any  $a_1, \dots, a_i, b_1, \dots, b_j \in A$ ,  $\bowtie_\sigma$  is defined recursively by

$$a_1 \bowtie_\sigma b_1 = a_1 \otimes b_1 + \sigma(a_1 \otimes b_1) + m(a_1 \otimes b_1),$$

$$\begin{aligned} a_1 \bowtie_\sigma (b_1 \otimes \dots \otimes b_j) &= a_1 \otimes b_1 \otimes \dots \otimes b_j + (\text{id}_A \otimes \bowtie_{\sigma(1,j-1)})(\beta_{1,1} \otimes \text{id}_A^{\otimes j-1})(a_1 \otimes b_1 \otimes \dots \otimes b_j) \\ &\quad + m(a_1 \otimes b_1) \otimes b_2 \otimes \dots \otimes b_j, \\ (a_1 \otimes \dots \otimes a_i) \bowtie_\sigma b_1 &= a_1 \otimes ((a_2 \otimes \dots \otimes a_i) \bowtie_\sigma b_1) + \beta_{i,1}(a_1 \otimes \dots \otimes a_i \otimes b_1) \\ &\quad + (m \otimes \text{id}_A^{\otimes i-1})(\text{id}_A \otimes \beta_{i-1,1})(a_1 \otimes \dots \otimes a_i \otimes b_1), \end{aligned}$$

and

$$\begin{aligned} (a_1 \otimes \dots \otimes a_i) \bowtie_\sigma (b_1 \otimes \dots \otimes b_j) &= a_1 \otimes ((a_2 \otimes \dots \otimes a_i) \bowtie_\sigma (b_1 \otimes \dots \otimes b_j)) \\ &\quad + (\text{id}_A \otimes \bowtie_{\sigma(i,j-1)})(\beta_{i,1} \otimes \text{id}_A^{\otimes j-1})(a_1 \otimes \dots \otimes a_i \otimes b_1 \otimes \dots \otimes b_j) \\ &\quad + (m \otimes \bowtie_{\sigma(i-1,j-1)})(\text{id}_A \otimes \beta_{i-1,1} \otimes \text{id}_A^{\otimes j-1})(a_1 \otimes \dots \otimes a_i \otimes b_1 \otimes \dots \otimes b_j), \end{aligned}$$

where  $\bowtie_{\sigma(i,j)}$  denotes the restriction of  $\bowtie_\sigma$  on  $A^{\otimes i} \otimes A^{\otimes j}$ .

*Remark 3.1.* 1. Given a braided algebra  $(A, m, \sigma)$ ,  $T_\sigma^{qsh}(A) = (T(A), \bowtie_\sigma)$  is an associative algebra with unit  $1 \in \mathbb{K}$ , and called the *quantum quasi-shuffle algebra* built on  $(A, m, \sigma)$ . Furthermore, the algebra  $T_\sigma^{qsh}(A)$ , together with the braiding  $\beta$  and the deconcatenation coproduct  $\delta$ , forms a braided bialgebra in the sense of [27] (see [13]). The set of primitive elements is exactly  $A$ .

2. By using Example 2.2, Proposition 17 in [14] can be applied to any vector space whose basis is at most countable. In other words, for any vector space  $V$  with at most countable basis, one can provide a linear basis of  $T(V)$  by combining the quantum quasi-shuffle product with Lyndon words.

**Example 3.2** (Hoffman's q-deformation). In [10], Hoffman defined his q-deformation of quasi-shuffles. It is an attempt to deform the quasi-shuffle product according to the quantum shuffle product. Now we give an explanation of the q-deformation from a point of view of quantum quasi-shuffles. Let  $X$  be a locally finite set, i.e.,

$X$  is a disjoint union of finite set  $X_n$ , whose elements are called letters of degree  $n$ , for  $n \geq 1$ . We denote by  $\mathfrak{X}$  the vector space spanned by  $X$ . The elements in  $T(\mathfrak{X})$ , which are of the form  $a_1 \otimes a_2 \otimes \cdots \otimes a_m$  with  $a_i \in X$ , are called words. Let  $[\cdot]$  be a graded associative product on  $\mathfrak{X}$ . Hoffman defined an associative product  $*_q$  on  $T(\mathfrak{X})$ : for any words  $w_1, w_2 \in T(\mathfrak{X})$  and letters  $a, b \in X$ ,

$$\begin{aligned} (a \otimes w_1) *_q (b \otimes w_2) &= a \otimes (w_1 *_q (b \otimes w_2)) + q^{|a \otimes w_1||b|} b \otimes ((a \otimes w_1) *_q w_2) \\ &\quad + q^{|w_1||b|} [a, b] \otimes (w_1 *_q w_2), \end{aligned}$$

where  $|w|$  is the degree of a word, i.e., the sum of degrees of its factors.

We define a braiding  $\sigma$  on  $\mathfrak{X}$  as follows: for  $x \in X_i$  and  $y \in X_j$ ,  $\sigma(x \otimes y) = q^{ij} y \otimes x$ , where  $q \in K$  is a nonzero scalar. Since the product  $[\cdot]$  preserves the grading, it is an easy exercise to verify that  $(\mathfrak{X}, [\cdot], \sigma)$  is a braided algebra. By comparing their reductive formulas, one can see that the quantum quasi-shuffle algebra built on  $(\mathfrak{X}, [\cdot], \sigma)$  is just Hoffman's  $q$ -deformation.

A similar discussion can be given to the formula of the multiplication rule of quantum quasi-monomial functions in [28].

#### 4. AN EXPLICIT FORMULA FOR THE QUANTUM QUASI-SHUFFLE PRODUCT

In order to give a more explicit description of the quantum quasi-shuffle product, we need to recall some terminologies introduced in [8]. An  $(i, j)$ -*shuffle* is an element  $w \in \mathfrak{S}_{i+j}$  such that  $w(1) < \cdots < w(i)$  and  $w(i+1) < \cdots < w(i+j)$ . We denote by  $\mathfrak{S}_{i,j}$  the set of all  $(i, j)$ -shuffles. Given an  $(i, j)$ -shuffle  $w$ , a pair  $(k, k+1)$ , where  $1 \leq k < i+j$ , is called an *admissible pair* for  $w$  if  $w^{-1}(k) \leq i < w^{-1}(k+1)$ . We denote by  $\mathcal{T}^w$  the set of all admissible pairs for  $w$ . For any subset  $S$  of  $\mathcal{T}^w$ , the pair  $(w, S)$  is called a *mixable  $(i, j)$ -shuffle*. We denote by  $\overline{\mathfrak{S}}_{i,j}$  the set of all mixable  $(i, j)$ -shuffles, i.e.,

$$\overline{\mathfrak{S}}_{i,j} = \{(w, S) | w \in \mathfrak{S}_{i,j}, S \subset \mathcal{T}^w\}.$$

Let  $(A, m, \sigma)$  be a braided algebra. Define  $m^k : A^{\otimes k+1} \rightarrow A$  recursively by  $m^0 = \text{id}_A$ ,  $m^1 = m$  and  $m^k = m(\text{id}_A \otimes m^{k-1})$  for  $k \geq 2$ . Given  $n \in \mathbb{N}$ , we denote

$$\mathcal{C}(n) = \{I = (i_1, \dots, i_k) \in \mathbb{N}^k | 1 \leq k \leq n, i_1 + \cdots + i_k = n\}.$$

The elements in  $\mathcal{C}(n)$  are called *compositions* of  $n$ . For any  $I = (i_1, \dots, i_k) \in \mathcal{C}(n)$ , we define  $m_I = m^{i_1-1} \otimes \cdots \otimes m^{i_k-1}$ . For any  $(w, S) \in \overline{\mathfrak{S}}_{i,j}$ , we associate to  $S$  a composition  $\text{cp}(S)$  of  $i+j$  as follows: if  $S = \{(k_1, k_1+1), \dots, (k_s, k_s+1)\}$  with  $k_1 < \cdots < k_s$ , set

$$\text{cp}(S) = (\underbrace{1, \dots, 1}_{k_1-1 \text{ copies}}, \underbrace{2, 1, \dots, 1}_{k_2-k_1-2 \text{ copies}}, \dots, \underbrace{2, 1, \dots, 1}_{i+j-k_s-1 \text{ copies}}).$$

By convention, we set  $\text{cp}(\emptyset) = (1, 1, \dots, 1)$ . Denote  $T_{(w,S)}^\sigma = m_{\text{cp}(S)} \circ T_w^\sigma$ .

**Theorem 4.1.** *Let  $(A, m, \sigma)$  be a braided algebra. Then for any  $a_1, \dots, a_{i+j} \in A$ ,*

$$(a_1 \otimes \cdots \otimes a_i) \rtimes_\sigma (a_{i+1} \otimes \cdots \otimes a_{i+j}) = \sum_{(w,S) \in \overline{\mathfrak{S}}_{i,j}} T_{(w,S)}^\sigma(a_1 \otimes \cdots \otimes a_{i+j}).$$

*Proof.* We use induction on  $i + j$ .

When  $i = j = 1$ ,

$$a_1 \bowtie_\sigma a_2 = m(a_1 \otimes a_2) + a_1 \otimes a_2 + \sigma(a_1 \otimes a_2).$$

On the other hand,  $\overline{\mathfrak{S}}_{1,1} = \{(1_{\mathfrak{S}_2}, \emptyset), (1_{\mathfrak{S}_2}, \{(1, 2)\}), (s_1, \emptyset)\}$ , where  $s_1$  is the generator of  $\mathfrak{S}_2$ . So the formula holds.

We assume that the formula is true in the case  $\leq i + j$ . By the inductive formula of quantum quasi-shuffles, we have that

$$\begin{aligned} & (a_1 \otimes \cdots \otimes a_{i+1}) \bowtie_\sigma (a_{i+2} \otimes \cdots \otimes a_{i+j+1}) \\ &= a_1 \otimes ((a_2 \otimes \cdots \otimes a_{i+1}) \bowtie_\sigma (a_{i+2} \otimes \cdots \otimes a_{i+j+1})) \\ & \quad + (\text{id}_A \otimes \bowtie_{\sigma(i+1, j-1)})(\beta_{i+1,1} \otimes \text{id}_A^{\otimes j-1})(a_1 \otimes \cdots \otimes a_{i+j+1}) \\ & \quad + (m \otimes \bowtie_{\sigma(i, j-1)})(\text{id}_A \otimes \beta_{i,1} \otimes \text{id}_A^{\otimes j-1})(a_1 \otimes \cdots \otimes a_{i+j+1}) \\ &= \sum_{(w, S) \in \overline{\mathfrak{S}}_{i,j}} (\text{id}_A \otimes T_{(w, S)}^\sigma)(a_1 \otimes \cdots \otimes a_{i+j+1}) \\ & \quad + \sum_{(w, S) \in \overline{\mathfrak{S}}_{i+1, j-1}} (\text{id}_A \otimes T_{(w, S)}^\sigma)(\beta_{i+1,1} \otimes \text{id}_A^{\otimes j-1})(a_1 \otimes \cdots \otimes a_{i+j+1}) \\ & \quad + \sum_{(w, S) \in \overline{\mathfrak{S}}_{i, j-1}} (m \otimes T_{(w, S)}^\sigma)(\text{id}_A \otimes \beta_{i,1} \otimes \text{id}_A^{\otimes j-1})(a_1 \otimes \cdots \otimes a_{i+j+1}) \\ &= \sum_{(w, S) \in \overline{\mathfrak{S}}_{i,j}} (\text{id}_A \otimes m_{\text{cp}(S)})T_{1_{\mathfrak{S}_1} \times w}^\sigma(a_1 \otimes \cdots \otimes a_{i+j+1}) \\ & \quad + \sum_{(w, S) \in \overline{\mathfrak{S}}_{i+1, j-1}} (\text{id}_A \otimes m_{\text{cp}(S)})T_{(1_{\mathfrak{S}_1} \times w) \circ (\chi_{i+1,1} \times 1_{\mathfrak{S}_{j-1}})}^\sigma(a_1 \otimes \cdots \otimes a_{i+j+1}) \\ & \quad + \sum_{(w, S) \in \overline{\mathfrak{S}}_{i, j-1}} (m \otimes m_{\text{cp}(S)})T_{(1_{\mathfrak{S}_2} \times w) \circ (1_{\mathfrak{S}_1} \times \chi_{i,1} \times 1_{\mathfrak{S}_{j-1}})}^\sigma(a_1 \otimes \cdots \otimes a_{i+j+1}), \end{aligned}$$

where the third equality follows from the fact that all the expressions of the permutations being lifted are reduced.

Denote

$$\begin{aligned} \mathcal{S}_1 &= \{(w, S) \in \overline{\mathfrak{S}}_{i+1, j} | (1, 2) \notin S, w(1) = 1\}, \\ \mathcal{S}_2 &= \{(w, S) \in \overline{\mathfrak{S}}_{i+1, j} | (1, 2) \notin S, w(i+2) = 1\}, \end{aligned}$$

and

$$\mathcal{S}_3 = \{(w, S) \in \overline{\mathfrak{S}}_{i+1, j} | (1, 2) \in S\}.$$

For any  $(i+1, j)$ -shuffle  $w$ , one has either  $w(1) = 1$  or  $w(i+2) = 1$ . Therefore  $\mathcal{S}_1$ ,  $\mathcal{S}_2$  and  $\mathcal{S}_3$  are mutually disjoint, and  $\mathfrak{S}_{i+1, j} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ .

We make a further observation. It is easy to see that there is a one-to-one correspondence between  $\mathfrak{S}_{i,j}$  and  $\{w \in \mathfrak{S}_{i+1, j} | w(1) = 1\}$  given by  $w \mapsto 1_{\mathfrak{S}_1} \times w$  for any  $w \in \mathfrak{S}_{i,j}$ . So

$$\mathcal{S}_1 = \{(1_{\mathfrak{S}_1} \times w, S) | w \in \mathfrak{S}_{i,j}, S \subset \mathcal{T}^{1_{\mathfrak{S}_1} \times w}, (1, 2) \notin S\}.$$

There is a one-to-one correspondence between  $\mathfrak{S}_{i+1, j-1}$  and  $\{w \in \mathfrak{S}_{i+1, j} | w(i+2) = 1\}$  given by  $w \mapsto \tilde{w} = (1_{\mathfrak{S}_1} \times w) \circ (\chi_{i+1,1} \times 1_{\mathfrak{S}_{j-1}})$  for any  $w \in \mathfrak{S}_{i+1, j-1}$ .



Consequently,

$$\mathcal{S}_2 = \{(\tilde{w}, S) | w \in \mathfrak{S}_{i+1, j-1}, S \subset \mathcal{T}^{\tilde{w}}, (1, 2) \notin S\}.$$

Finally, for any  $(w, S) \in \mathcal{S}_3$ , we must have that  $w(1) = 1$  and  $w(i+2) = 2$ . There is a one-to-one correspondence between  $\mathfrak{S}_{i, j-1}$  and  $\{w \in \mathfrak{S}_{i+1, j} | w(1) = 1, w(i+2) = 2\}$  given by  $w \mapsto \bar{w} = (1_{\mathfrak{S}_2} \times w) \circ (1_{\mathfrak{S}_1} \times \chi_{i,1} \times 1_{\mathfrak{S}_{j-1}})$  for any  $w \in \mathfrak{S}_{i, j-1}$ . So

$$\mathcal{S}_3 = \{(\bar{w}, S) \in \overline{\mathfrak{S}}_{i+1, j} | w \in \mathfrak{S}_{i, j-1}, (1, 2) \in S\}.$$

As a conclusion, the three terms in the final step of the preceding computation come from  $\mathcal{S}_1$ ,  $\mathcal{S}_2$  and  $\mathcal{S}_3$  respectively. So we have that

$$(a_1 \otimes \cdots \otimes a_{i+1}) \rtimes_{\sigma} (a_{i+2} \otimes \cdots \otimes v_{i+j+1}) = \sum_{(w, S) \in \overline{\mathfrak{S}}_{i+1, j}} T_{(w, S)}^{\sigma}(a_1 \otimes \cdots \otimes a_{i+j+1}),$$

which completes the induction.  $\square$

*Remark 4.2.* Let  $(A, m)$  be an algebra and  $\lambda$  be a scalar in  $\mathbb{K}$ . Then  $(A, \lambda m)$  becomes a braided algebra with respect to the usual flip map. In this case, the formula in Theorem 4.1 coincides with the one of mixable shuffle product introduced in [8].

## 5. THE SUBALGEBRA GENERATED BY PRIMITIVE ELEMENTS

Assume again that  $(A, m, \sigma)$  is a braided algebra. We denote by  $S_{\sigma}^{qsh}(A)$  the subalgebra of  $T_{\sigma}^{qsh}(A)$  generated by  $A$ . To describe this subalgebra, we need to introduce some notation.

For a fixed  $n \in \mathbb{N}$  and any  $w \in \mathfrak{S}_n$ , we denote

$$\mathcal{S}^w = \{(k, k+1) | 1 \leq k < n, w^{-1}(k) < w^{-1}(k+1)\},$$

and

$$\overline{\mathfrak{S}}_n = \{(w, S) | w \in \mathfrak{S}_n, S \subset \mathcal{S}^w\}.$$

For any  $(w, S) \in \overline{\mathfrak{S}}_n$ , we associate to  $S$  a composition  $\text{cp}(S)$  of  $n$  as follows. Let  $S = \{(k_1, k_1+1), \dots, (k_s, k_s+1)\}$  with  $k_1 < \dots < k_s$ . We divide  $\{k_1, \dots, k_s\}$  into several subsets

$$\{k_1, \dots, k_{i_1}\}, \{k_{i_1+1}, \dots, k_{i_1+i_2}\}, \dots, \{k_{i_1+\dots+i_{r-1}+1}, \dots, k_s\},$$

which obey the rule that:

$$\begin{cases} k_1 + 1 = k_2, k_2 + 1 = k_3, \dots, k_{i_1-1} + 1 = k_{i_1}, \\ k_{i_1+1} + 1 = k_{i_1+2}, k_{i_1+2} + 1 = k_{i_1+3}, \dots, k_{i_1+i_2-1} + 1 = k_{i_2}, \\ \dots, \\ k_{i_1+\dots+i_{r-1}+1} + 1 = k_{i_1+\dots+i_{r-1}+2}, \dots, k_{s-1} + 1 = k_s, \end{cases}$$

but  $k_{i_1} + 1 < k_{i_1+1}, k_{i_1+i_2} + 1 < k_{i_1+i_2+1}, \dots, k_{i_1+\dots+i_{r-1}} + 1 < k_{i_1+\dots+i_{r-1}+1}$ . Denote  $i_r = s - i_1 - \dots - i_{r-1}$ . Then we write

$$\text{cp}(S) = (\underbrace{1, \dots, 1}_{k_1-1 \text{ copies}}, \underbrace{i_1+1, 1, \dots, 1}_{k_{i_1+1}-k_{i_1}-1 \text{ copies}}, i_2+1, \dots, i_r+1, \underbrace{1, \dots, 1}_{n-k_{i_r}-1 \text{ copies}}).$$

We define as before the map  $T_{(w, S)}^{\sigma} = m_{\text{cp}(S)} \circ T_w^{\sigma}$  for any  $(w, S) \in \overline{\mathfrak{S}}_n$ .

Now we provide a decomposition of  $\overline{\mathfrak{S}}_{n+1}$  which will be used later. For any  $1 \leq i \leq n+1$ , we denote  $\mathfrak{S}_{n+1}(i) = \{w \in \mathfrak{S}_{n+1} | w(1) = i\}$ . It is clear that  $\mathfrak{S}_{n+1}$  is the

disjoint union of all  $\mathfrak{S}_{n+1}(i)$ 's, and for each  $i$  there is a one-to-one correspondence between  $\mathfrak{S}_n$  and  $\mathfrak{S}_{n+1}(i)$  given by  $w \mapsto L(w, i) = (\chi_{1,i-1} \times 1_{\mathfrak{S}_{n+1-i}}) \circ (1_{\mathfrak{S}_1} \times w)$  for any  $w \in \mathfrak{S}_{n-1}$ . So

$$\begin{aligned} \mathfrak{S}_{n+1} &= \bigcup_{i=1}^{n+1} \mathfrak{S}_{n+1}(i) \\ &= \bigcup_{i=1}^{n+1} \bigcup_{w \in \mathfrak{S}_n} \{L(w, i)\} \\ &= \bigcup_{w \in \mathfrak{S}_n} \bigcup_{i=1}^{n+1} \{L(w, i)\}. \end{aligned}$$

Then we have that

$$\begin{aligned} \overline{\mathfrak{S}}_{n+1} &= \bigcup_{w \in \mathfrak{S}_n} \bigcup_{i=1}^{n+1} \{(L(w, i), S) | S \subset \mathcal{S}^{L(w, i)}\} \\ &= \left( \bigcup_{w \in \mathfrak{S}_n} \bigcup_{i=1}^{n+1} \{(L(w, i), S) | S \subset \mathcal{S}^{L(w, i)}, (i, i+1) \notin S\} \right) \\ &\quad \cup \left( \bigcup_{w \in \mathfrak{S}_n} \bigcup_{i=1}^{n+1} \{(L(w, i), S) | S \subset \mathcal{S}^{L(w, i)}, (i, i+1) \in S\} \right). \end{aligned}$$

All the unions above are disjoint.

Given  $w \in \mathfrak{S}_n$  and  $S \subset \mathcal{S}^w$  with  $\text{cp}(S) = (i_1, \dots, i_s)$ . For any  $0 \leq k \leq s$ , we denote

$$\begin{aligned} \text{cp}(S)_k &= (i_1, \dots, i_k, 1, i_{k+1}, \dots, i_s), \\ \text{cp}(S)^k &= (i_1, \dots, i_{k-1}, i_k + 1, i_{k+1}, \dots, i_s), \end{aligned}$$

and

$$I_k = (\underbrace{1, \dots, 1}_{k \text{ copies}}, 2, \underbrace{1, \dots, 1}_{n-1-|S|-k \text{ copies}}).$$

Here,  $|S|$  denotes the cardinality of the set  $S$ .

**Lemma 5.1.** *Under the assumptions above, we have*

$$\begin{aligned} (\beta_{1,k} \otimes \text{id}_A^{\otimes n-|S|-k})(\text{id}_A \otimes T_{(w,S)}^\sigma) &= m_{\text{cp}(S)_k} T_{L(w, i_1 + \dots + i_k + 1)}, \\ m_{I_k}(\beta_{1,k} \otimes \text{id}_A^{\otimes n-|S|-k})(\text{id}_A \otimes T_{(w,S)}^\sigma) &= m_{\text{cp}(S)^k} T_{L(w, i_1 + \dots + i_k + 1)}. \end{aligned}$$

*Proof.* Since  $\sigma(\text{id}_A \otimes m^l) = (m^l \otimes \text{id}_A)\beta_{1,l+1}$  for any  $l$  (see Lemma 2 in [2]),

$$\begin{aligned} &(\beta_{1,k} \otimes \text{id}_A^{\otimes n-|S|-k})(\text{id}_A \otimes T_{(w,S)}^\sigma) \\ &= (\beta_{1,k} \otimes \text{id}_A^{\otimes n-|S|-k})(\text{id}_A \otimes m_{\text{cp}(S)})(\text{id}_A \otimes T_w^\sigma) \\ &= m_{\text{cp}(S)_k}(\beta_{1, i_1 + \dots + i_k} \otimes \text{id}_A^{\otimes i_{k+1} + \dots + i_s})(\text{id}_A \otimes T_w^\sigma) \\ &= m_{\text{cp}(S)_k} T_{(\chi_{1, i_1 + \dots + i_k} \times 1_{\mathfrak{S}_{i_{k+1} + \dots + i_s}}) \circ (1_{\mathfrak{S}_1} \times w)}^\sigma \end{aligned}$$

$$= m_{\text{cp}(S)_k} T_{L(w, i_1 + \dots + i_k + 1)}.$$

The second equality is a consequence of the first one.  $\square$

**Theorem 5.2.** *Let  $(A, m, \sigma)$  be a braided algebra. For any  $a_1, \dots, a_n \in A$ , we have that*

$$a_1 \bowtie_\sigma \dots \bowtie_\sigma a_n = \sum_{(w, S) \in \overline{\mathfrak{S}}_n} T_{(w, S)}^\sigma(a_1 \otimes \dots \otimes a_n).$$

Therefore  $S_\sigma^{qsh}(A) = \sum_{n \geq 0} \text{Im}(\sum_{(w, S) \in \overline{\mathfrak{S}}_n} T_{(w, S)}^\sigma)$ .

*Proof.* We use induction on  $n$ .

When  $n = 2$ , it is trivial since

$$\overline{\mathfrak{S}}_2 = \{(1_{\mathfrak{S}_2}, \emptyset), (1_{\mathfrak{S}_2}, \{(1, 2)\}), (s_1, \emptyset)\}.$$

By Theorem 3.5, we have that for any  $a_1, \dots, a_{r+1} \in A$ ,

$$\begin{aligned} & a_1 \bowtie_\sigma (a_2 \otimes \dots \otimes a_{r+1}) \\ &= \sum_{k=0}^r (\beta_{1,k} \otimes \text{id}_A^{\otimes r-k})(a_1 \otimes \dots \otimes a_{r+1}) \\ & \quad + \sum_{k=0}^{r-1} (\text{id}_A^{\otimes k} \otimes m \otimes \text{id}_A^{\otimes r-k-1})(\beta_{1,k} \otimes \text{id}_A^{\otimes r-k})(a_1 \otimes \dots \otimes a_{r+1}). \end{aligned}$$

Therefore,

$$\begin{aligned} & a_1 \bowtie_\sigma \dots \bowtie_\sigma a_{n+1} \\ &= a_1 \bowtie_\sigma \left( \sum_{(w, S) \in \overline{\mathfrak{S}}_n} T_{(w, S)}^\sigma(a_2 \otimes \dots \otimes a_{n+1}) \right) \\ &= \sum_{(w, S) \in \overline{\mathfrak{S}}_n} \sum_{k=0}^{n-|S|} (\beta_{1,k} \otimes \text{id}_A^{\otimes n-|S|-k})(a_1 \otimes T_{(w, S)}^\sigma(a_2 \otimes \dots \otimes a_{n+1})) \\ & \quad + \sum_{(w, S) \in \overline{\mathfrak{S}}_n} \sum_{k=0}^{n-|S|-1} (\text{id}_A^{\otimes k} \otimes m \otimes \text{id}_A^{\otimes n-|S|-k-1}) \\ & \quad \quad \quad \circ (\beta_{1,k} \otimes \text{id}_A^{\otimes n-|S|-k})(a_1 \otimes T_{(w, S)}^\sigma(a_2 \otimes \dots \otimes a_{n+1})) \\ &= \sum_{(w, S) \in \overline{\mathfrak{S}}_n} \sum_{k=0}^{n-|S|} m_{\text{cp}(S)_k} T_{L(w, i_1 + \dots + i_k + 1)}^\sigma(a_1 \otimes \dots \otimes a_{n+1}) \\ & \quad + \sum_{(w, S) \in \overline{\mathfrak{S}}_n} \sum_{k=0}^{n-|S|-1} m_{\text{cp}(S)_k} T_{L(w, i_1 + \dots + i_k + 1)}^\sigma(a_1 \otimes \dots \otimes a_{n+1}), \end{aligned}$$

where the last equality follows from the preceding lemma.

On the other hand, by the decomposition of  $\overline{\mathfrak{S}}_{n+1}$  mentioned before,

$$\begin{aligned} & \sum_{(w,S) \in \overline{\mathfrak{S}}_{n+1}} T_{(w,S)}^\sigma(a_1 \otimes \cdots \otimes a_{n+1}) \\ &= \sum_{w \in \mathfrak{S}_n} \sum_{i=1}^{n+1} \sum_{\substack{S \subset \mathcal{S}^{L(w,i)} \\ (i,i+1) \notin S}} T_{(L(w,i),S)}^\sigma(a_1 \otimes \cdots \otimes a_{n+1}) \\ &+ \sum_{w \in \mathfrak{S}_n} \sum_{i=1}^{n+1} \sum_{\substack{S \subset \mathcal{S}^{L(w,i)} \\ (i,i+1) \in S}} T_{(L(w,i),S)}^\sigma(a_1 \otimes \cdots \otimes a_{n+1}). \end{aligned}$$

We compare the terms in these two expressions. Notice that for a fixed  $1 \leq i \leq n+1$  and  $S \subset \mathcal{S}^{L(w,i)}$  with  $(i, i+1) \notin S$ , there is a unique  $S' \in \mathcal{S}^w$  such that  $\text{cp}(S) = \text{cp}(S')_i$ . Indeed, we can write down  $S'$  explicitly: if  $S = \{(k_1, k_1 + 1), \dots, (k_s, k_s + 1)\}$  with  $k_1 < \cdots < k_l < i < k_{l+1} < \cdots < k_s$ , then  $S' = \{(k_1, k_1 + 1), \dots, (k_l, k_l + 1), (k_{l+1} - 1, k_{l+1}), \dots, (k_s - 1, k_s)\}$ . Similarly, if  $(i, i+1) \in S$ , there is a unique  $S'' \in \mathcal{S}^w$  such that  $\text{cp}(S) = \text{cp}(S'')^i$ . It follows that every term in  $\sum_{(w,S) \in \overline{\mathfrak{S}}_{n+1}} T_{(w,S)}^\sigma(a_1 \otimes \cdots \otimes a_{n+1})$  is from exactly one term in the formula of  $a_1 \bowtie_\sigma \cdots \bowtie_\sigma a_{n+1}$ . The converse is also true. Since all terms in each formula are mutually distinct, we get the conclusion.  $\square$

*Remark 5.3.* Consider Example 3.2 and let  $(\mathfrak{X}, [\cdot, \cdot], \sigma)$  be the braided algebra introduced there. For any  $w \in \mathfrak{S}_n$ , we denote  $\iota(w) = \{(i, j) | 1 \leq i < j \leq n, w(i) > w(j)\}$ . Then for any  $a_1, \dots, a_n \in X$ ,

$$T_w^\sigma(a_1 \otimes \cdots \otimes a_n) = q^{\sum_{(i,j) \in \iota(w)} |a_i||a_j|} a_{w^{-1}(1)} \otimes \cdots \otimes a_{w^{-1}(n)}.$$

For any two compositions  $I = (i_1, \dots, i_k)$  and  $J = (j_1, \dots, j_l)$  of  $n$ , we say  $I$  is a *refinement* of  $J$ , written by  $I \succeq J$ , if there are  $r_1, \dots, r_l \in \mathbb{N}$  such that  $r_1 + \cdots + r_l = k$  and

$$i_1 + \cdots + i_{r_1} = j_1, i_{r_1+1} + \cdots + i_{r_1+r_2} = j_2, \dots, i_{r_1+\cdots+r_{l-1}+1} + \cdots + i_k = j_l.$$

For instance,  $(1, 2, 2, 3) \succeq (3, 2, 3)$ . For any  $w \in \mathfrak{S}_n$ , let  $C(w)$  be the composition  $(i_1, \dots, i_k)$  of  $n$  such that

$$\{i_1, i_1 + i_2, \dots, i_1 + \cdots + i_{k-1}\} = \{l | 1 \leq l \leq n-1, w(l) > w(l+1)\}.$$

For any  $I = (i_1, \dots, i_l) \in \mathcal{C}(n)$ , we write  $I[a_1 \otimes \cdots \otimes a_n] = [\cdot]_I(a_1 \otimes \cdots \otimes a_n)$ .

Observing that for any  $w \in \mathfrak{S}_n$  there is a one-to-one correspondence between  $S \subset \mathcal{S}^w$  and  $I \in \mathcal{C}(n)$  with  $I \succeq C(w)$ , one has immediately that

$$a_1 * q \cdots * q a_n = \sum_{w \in \mathfrak{S}_n} q^{\sum_{(i,j) \in \iota(w)} |a_i||a_j|} \sum_{I \succeq C(w)} I[a_{w^{-1}(1)} \otimes \cdots \otimes a_{w^{-1}(n)}].$$

This formula is given by Hoffman when  $[\cdot, \cdot]$  is commutative (see Lemma 5.2 in [10]).

We conclude this section by an interesting formula. Let  $V$  be a vector space with basis  $\{e_i\}_{i \in \mathbb{N}}$ , and  $(V, m, \sigma)$  be the braided algebra structure given in Example 2.3. So we have that  $\sigma(e_i \otimes e_j) = q_{ij} e_j \otimes e_i$  and  $m(e_i \otimes e_j) = e_{i+j}$ .

**Proposition 5.4.** *For any  $i, k \in \mathbb{N}$ , we have that*

$$e_i^{\bowtie_\sigma k} = \sum_{n=1}^k \sum_{\substack{l_1 + \dots + l_n = k \\ 1 \leq l_1, \dots, l_n \leq k}} \frac{[k]_{q_{ii}}!}{[l_1]_{q_{ii}}! \dots [l_n]_{q_{ii}}!} e_{l_1 i} \otimes \dots \otimes e_{l_n i}.$$

*Proof.* It is a direct verification by using induction and Theorem 3.6 or Hoffman's formula.  $\square$

## 6. THE DUAL CONSTRUCTION

In this section, we give a dual construction of the quantum quasi-shuffle algebra by using the universal property of tensor algebras. First of all, we study a special coalgebra structure on  $T(C)$  which is a generation of the quantized cofree coalgebra structure. For coalgebras, we adopt Sweedler's notation. That means for a coalgebra  $(C, \Delta, \varepsilon)$  and any  $c \in C$ , we denote

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)},$$

or simply,  $\Delta(c) = c_{(1)} \otimes c_{(2)}$ .

To extend algebra structures to quantized case, one needs the notion of braided algebras. By contrast, in the case of coalgebras, one needs the so-called braided coalgebras.

**Definition 6.1.** Let  $C = (C, \Delta, \varepsilon)$  be a coalgebra with coproduct  $\Delta$  and counit  $\varepsilon$ , and  $\sigma$  be a braiding on  $C$ . We call  $(C, \Delta, \sigma)$  a *braided coalgebra* if it satisfies the following conditions:

$$\begin{aligned} (\text{id}_C \otimes \Delta)\sigma &= \sigma_1 \sigma_2 (\Delta \otimes \text{id}_C), \\ (\Delta \otimes \text{id}_C)\sigma &= \sigma_2 \sigma_1 (\text{id}_C \otimes \Delta). \end{aligned}$$

In order to get new braided algebras and coalgebras from old ones, we need the proposition below.

**Proposition 6.2** ([9], Proposition 4.2). *1. For a braided algebra  $(A, \mu, \sigma)$  and any  $i \in \mathbb{N}$ ,  $(A^{\otimes i}, \mu_{\sigma, i}, \beta_{ii})$  becomes a braided algebra with product  $\mu_{\sigma, i} = \mu^{\otimes i} \circ T_{w_i}^\sigma$ , where  $w_i \in \mathfrak{S}_{2i}$  is given by*

$$w_i = \begin{pmatrix} 1 & 2 & 3 & \dots & i & i+1 & i+2 & \dots & 2i \\ 1 & 3 & 5 & \dots & 2i-1 & 2 & 4 & \dots & 2i \end{pmatrix}.$$

*2. For a braided coalgebra  $(C, \Delta, \sigma)$ ,  $(C^{\otimes i}, \Delta_{\sigma, i}, \beta_{ii})$  becomes a braided coalgebra with coproduct  $\Delta_{\sigma, i} = T_{w_i}^\sigma \circ \Delta^{\otimes i}$  and counit  $\varepsilon^{\otimes i} : C^{\otimes i} \rightarrow \mathbb{K}^{\otimes i} \simeq \mathbb{K}$ .*

Let  $(C, \Delta, \sigma)$  be a braided coalgebra. Consider the tensor algebra  $T(C)$  with the concatenation product  $m$ . Then by the above proposition,  $\mathcal{T}_\beta^2(C) = (T(C) \underline{\otimes} T(C), m_{\beta, 2})$  and  $\mathcal{T}_\beta^3 = (T(C) \underline{\otimes} T(C) \underline{\otimes} T(C), m_{\beta, 3})$  are associative algebras.

We define

$$\begin{aligned} \phi_1 : C &\rightarrow T(C) \underline{\otimes} T(C), \\ c &\mapsto 1 \underline{\otimes} c + c \underline{\otimes} 1. \end{aligned}$$

By the universal property of tensor algebras, there exists an algebra map  $\Phi_1 : T(C) \rightarrow \mathcal{T}_\beta^2(C)$  whose restriction on  $C$  is  $\phi_1$ . Moreover  $\Phi_1$  is coassociative and is the dual of quantum shuffle product (see, e.g., [4]).

Now we define

$$\begin{aligned}\phi_2 : C &\rightarrow T(C) \otimes T(C), \\ c &\mapsto \sum c_{(1)} \otimes c_{(2)}.\end{aligned}$$

By the universal property of tensor algebras again, there exists an algebra map  $\Phi_2 : T(C) \rightarrow \mathcal{T}_\beta^2(C)$  whose restriction on  $C$  is  $\phi_2$ .

**Proposition 6.3.** *For any  $i \in \mathbb{N}$ , we have that  $\Phi_2|_{C^{\otimes i}} = \Delta_{\sigma, i}$ . So  $(T(C), \Phi_2, \beta)$  is a braided coalgebra.*

*Proof.* We use induction on  $i$ . When  $i = 1$ , it is trivial. Assume the equality holds for the case  $i < n$ . Then for any  $c_1, \dots, c_n \in C$ , we have

$$\begin{aligned}\Phi_2(c_1 \otimes \dots \otimes c_n) &= m_{\beta, 2}(\Phi_2(c_1) \otimes \Phi_2(c_2 \otimes \dots \otimes c_n)) \\ &= (\text{id}_{T(C)} \otimes \beta \otimes \text{id}_{T(C)})(\Delta(c_1) \otimes T_{w_{n-1}}^\sigma \circ \Delta^{\otimes n-1}(c_2 \otimes \dots \otimes c_n)) \\ &= (\text{id}_C \otimes T_{\chi_{1, n-1}}^\sigma \otimes \text{id}_C^{\otimes n-1})(\text{id}_C^{\otimes 2} \otimes T_{w_{n-1}}^\sigma) \Delta^{\otimes n}(c_1 \otimes \dots \otimes c_n) \\ &= T_{w_n}^\sigma \circ \Delta^{\otimes n}(c_2 \otimes \dots \otimes c_n),\end{aligned}$$

where the last equality follows from the fact that  $(1_{\mathfrak{G}_1} \times \chi_{1, n-1} \times 1_{\mathfrak{G}_{n-1}})(1_{\mathfrak{G}_2} \times w_{n-1}^{-1}) = w_n^{-1}$  and the expression is reduced.  $\square$

Let  $\phi = \phi_1 + \phi_2 : C \rightarrow \mathcal{T}_\beta^2(C)$  and  $\Phi$  be the algebraic map induced by the universal property of tensor algebras which extends  $\phi$ .

**Proposition 6.4.** *Under the notation above, the triple  $(T(C), \Phi, \beta)$  is a braided coalgebra.*

*Proof.* We first show that

$$\begin{cases} \beta_1 \beta_2 (\Phi \otimes \text{id}_{T(C)}) &= (\text{id}_{T(C)} \otimes \Phi) \beta, \\ \beta_2 \beta_1 (\text{id}_{T(C)} \otimes \Phi) &= (\Phi \otimes \text{id}_{T(C)}) \beta. \end{cases}$$

For any  $x \in C^{\otimes i}$  and  $y \in C^{\otimes j}$  we verify the first one on  $x \underline{\otimes} y$ . The second one can be verified similarly. We use induction on  $i$ .

When  $i = 1$ ,

$$\begin{aligned}\beta_1 \beta_2 (\Phi \otimes \text{id}_{T(C)})(x \underline{\otimes} y) &= \beta_1 \beta_2 (\phi_1 \otimes \text{id} + \phi_2 \otimes \text{id})(x \underline{\otimes} y) \\ &= (\text{id} \otimes \phi_1 + \text{id} \otimes \phi_2) \beta(x \underline{\otimes} y) \\ &= (\text{id}_{T(C)} \otimes \Phi) \beta(x \underline{\otimes} y).\end{aligned}$$

For any  $c \in C$ , we have

$$\begin{aligned}\beta_1 \beta_2 (\Phi \otimes \text{id}_{T(C)})(c \otimes x \underline{\otimes} y) &= \beta_1 \beta_2 \beta_3 \beta_4 \beta_2 (\Phi \otimes \Phi \otimes \text{id}_{T(C)})(c \underline{\otimes} x \underline{\otimes} y) \\ &= \beta_1 \beta_3 \beta_2 \beta_3 \beta_4 (\Phi \otimes \Phi \otimes \text{id}_{T(C)})(c \underline{\otimes} x \underline{\otimes} y)\end{aligned}$$

$$\begin{aligned}
 &= \beta_1 \beta_3 \beta_2 (\Phi \otimes \beta_1 \beta_2 (\Phi \otimes \text{id}_{T(C)})) (c \underline{\otimes} x \underline{\otimes} y) \\
 &= \beta_1 \beta_3 \beta_2 (\Phi \otimes \text{id}_{T(C)} \otimes \Phi) \beta_2 (c \underline{\otimes} x \underline{\otimes} y) \\
 &= \beta_3 (\beta_1 \beta_2 (\Phi \otimes \text{id}_{T(C)}) \otimes \Phi) \beta_2 (c \underline{\otimes} x \underline{\otimes} y) \\
 &= \beta_3 (\text{id}_{T(C)} \otimes \Phi \otimes \Phi) \beta_1 \beta_2 (c \underline{\otimes} x \underline{\otimes} y) \\
 &= (\text{id}_{T(C)} \otimes \Phi) \beta ((c \otimes x) \underline{\otimes} y).
 \end{aligned}$$

The next step is to show  $(\Phi \otimes \text{id}_{T(C)})\Phi = (\text{id}_{T(C)} \otimes \Phi)\Phi$ . Notice that for any  $c \in C$ ,

$$\begin{aligned}
 &(\Phi \otimes \text{id}_{T(C)})\Phi(c) \\
 &= (\Phi \otimes \text{id}_{T(C)})(c_{(1)} \underline{\otimes} c_{(2)} + 1 \underline{\otimes} c + c \underline{\otimes} 1) \\
 &= c_{(1)} \underline{\otimes} c_{(2)} \underline{\otimes} c_{(3)} + 1 \underline{\otimes} c_{(1)} \underline{\otimes} c_{(2)} + c_{(1)} \underline{\otimes} c_{(2)} \underline{\otimes} 1 \\
 &\quad + c_{(1)} \underline{\otimes} 1 \underline{\otimes} c_{(2)} + 1 \underline{\otimes} 1 \underline{\otimes} c + c_{(1)} \underline{\otimes} c_{(2)} \underline{\otimes} 1 + 1 \underline{\otimes} c \underline{\otimes} 1 + c \underline{\otimes} 1 \underline{\otimes} 1 \\
 &= (\text{id}_{T(C)} \otimes \Phi)\Phi(c).
 \end{aligned}$$

By the uniqueness of the universal property of  $T(C)$ , we only need to show that both  $(\Phi \otimes \text{id}_{T(C)})\Phi$  and  $(\text{id}_{T(C)} \otimes \Phi)\Phi$  are algebra morphisms from  $T(C)$  to  $\mathcal{T}_\beta^3$ . Since  $\Phi : T(C) \rightarrow \mathcal{T}_\beta^2(C)$  is an algebra morphism, we have that

$$\Phi \circ m = (m \otimes m)(\text{id}_{T(C)} \otimes \beta \otimes \text{id}_{T(C)})(\Phi \otimes \Phi).$$

So

$$\begin{aligned}
 &(\Phi \otimes \text{id}_{T(C)}) \circ \Phi \circ m \\
 &= (\Phi \otimes \text{id}_{T(C)})(m \otimes m)(\text{id}_{T(C)} \otimes \beta \otimes \text{id}_{T(C)})(\Phi \otimes \Phi) \\
 &= ((\Phi \circ m) \otimes m)(\text{id}_{T(C)} \otimes \beta \otimes \text{id}_{T(C)})(\Phi \otimes \Phi) \\
 &= (m \otimes m \otimes m)(\text{id}_{T(C)} \otimes \beta \otimes \text{id}_{T(C)} \otimes \text{id}_{T(C)} \otimes \text{id}_{T(C)}) \\
 &\quad \circ (\Phi \otimes \Phi \otimes \text{id}_{T(C)} \otimes \text{id}_{T(C)})(\text{id}_{T(C)} \otimes \beta \otimes \text{id}_{T(C)})(\Phi \otimes \Phi) \\
 &= (m \otimes m \otimes m) \beta_2 \\
 &\quad \circ (\text{id}_{T(C)} \otimes \text{id}_{T(C)} \otimes (\Phi \otimes \text{id}_{T(C)}) \beta \otimes \text{id}_{T(C)}) ((\Phi \otimes \text{id}_{T(C)}) \Phi \otimes \Phi) \\
 &= (m \otimes m \otimes m) \beta_2 \beta_4 \beta_3 \\
 &\quad \circ (\text{id}_{T(C)} \otimes \text{id}_{T(C)} \otimes \text{id}_{T(C)} \otimes \Phi \otimes \text{id}_{T(C)}) ((\Phi \otimes \text{id}_{T(C)}) \Phi \otimes \Phi) \\
 &= m_{\beta,3} ((\Phi \otimes \text{id}_{T(C)}) \Phi \otimes (\Phi \otimes \text{id}_{T(C)}) \Phi).
 \end{aligned}$$

It follows that  $(\Phi \otimes \text{id}_{T(C)})\Phi$  is an algebra morphism. Similarly, the map  $(\text{id}_{T(C)} \otimes \Phi)\Phi$  is also an algebra morphism.  $\square$

Now we begin to study the relation between the braided coalgebra  $(T(C), \Phi, \beta)$  and the quantum quasi-shuffle algebra. We show that they are dual to each other in the following sense.

Let  $\langle, \rangle : V \times W \rightarrow \mathbb{K}$  and  $\langle, \rangle' : V' \times W' \rightarrow \mathbb{K}$  be two bilinear non-degenerate forms. For any  $f \in \text{Hom}(V, V')$ , the adjoint operator  $\text{adj}(f) \in \text{Hom}(W', W)$  of  $f$  is

defined to be the one such that  $\langle x, \text{adj}(f)(y) \rangle = \langle f(x), y \rangle'$  for any  $x \in V$  and  $y \in W'$ . It is clear that  $\text{adj}(f \circ g) = \text{adj}(g) \circ \text{adj}(f)$ .

*Remark 6.5.* If there is a non-degenerate bilinear form between two vector spaces  $A$  and  $C$ , and  $(A, m, \sigma)$  is a braided algebra, then  $(C, \text{adj}(m), \text{adj}(\sigma))$  is a braided coalgebra. The converse is also true.

From now on, we assume that there always exists a non-degenerate bilinear form  $\langle, \rangle$  between two vector spaces  $A$  and  $C$ . It can be extended to a bilinear form  $\langle, \rangle: A^{\otimes n} \times C^{\otimes n} \rightarrow \mathbb{K}$  for any  $n \geq 1$  in the usual way: for any  $a_1, \dots, a_n \in A$  and  $c_1, \dots, c_n \in C$ ,

$$\langle a_1 \otimes \dots \otimes a_n, c_1 \otimes \dots \otimes c_n \rangle = \prod_{i=1}^n \langle a_i, c_i \rangle.$$

It induces a non-degenerate bilinear form  $\langle, \rangle: T(A) \times T(C) \rightarrow \mathbb{K}$  by setting that  $\langle x, y \rangle = 0$  for any  $x \in A^{\otimes i}$ ,  $y \in C^{\otimes j}$  and  $i \neq j$ . Then we can define a non-degenerate bilinear form  $\langle, \rangle: T(A) \underline{\otimes} T(A) \times T(C) \underline{\otimes} T(C) \rightarrow \mathbb{K}$  by requiring that  $\langle u \underline{\otimes} v, x \underline{\otimes} y \rangle = \langle u, x \rangle \langle v, y \rangle$  for any  $u, v \in T(A)$  and  $x, y \in T(C)$ .

If  $(C, \Delta, \sigma)$  is a braided coalgebra, we denote  $\tau = \text{adj}(\sigma)$  and  $\alpha = \text{adj}(\beta)$ . Then  $\alpha$  is a braiding on  $T(A)$  and  $\alpha_{i,j} = T_{\chi_{ji}}^\tau = T_{\chi_{ij}}$ .

**Theorem 6.6.** *Under the assumptions above, we have that  $\text{adj}(\Phi) = \bowtie_\sigma$ .*

*Proof.* For any  $c_1, \dots, c_n \in C$ , we notice that

$$\begin{aligned} \Phi(c_1 \otimes \dots \otimes c_n) &= m_{\beta,2}(\Phi(c_1) \underline{\otimes} \Phi(c_2 \otimes \dots \otimes c_n)) \\ &= m_{\beta,2}((1 \underline{\otimes} c_1) \underline{\otimes} \Phi(c_2 \otimes \dots \otimes c_n)) \\ &\quad + m_{\beta,2}((v_1 \underline{\otimes} 1) \underline{\otimes} \Phi(c_2 \otimes \dots \otimes c_n)) \\ &\quad + m_{\beta,2}((c_1(1) \underline{\otimes} c_1(2)) \underline{\otimes} \Phi(c_2 \otimes \dots \otimes c_n)) \\ &= (\beta_{1,?} \otimes \text{id}_C^{\otimes n-1-?})(\text{id}_C \otimes \Phi)(c_1 \otimes \dots \otimes c_n) \\ &\quad + (\text{id}_C \otimes \Phi)(c_1 \otimes \dots \otimes c_n) \\ &\quad + (\text{id}_C \otimes \beta_{1,?} \otimes \text{id}_C^{\otimes n-1-?})(\Delta \otimes \Phi)(c_1 \otimes \dots \otimes c_n). \end{aligned}$$

Here, since  $\Phi(C^{\otimes k}) \subset C^{\otimes 0} \underline{\otimes} C^{\otimes k} + C^{\otimes k} \underline{\otimes} C^{\otimes 0} + \dots + C^{\otimes k} \underline{\otimes} C^{\otimes k}$ , we denote by  $\beta_{1,?}$  the action of  $\beta$  on  $C \underline{\otimes} C^{\otimes ?}$  where  $C^{\otimes ?}$  is the left factor of some component in  $\Phi(C^{\otimes k})$ .

We denote by  $\text{adj}(\Phi)_{(k,l)}$  the action of  $\text{adj}(\Phi)$  on  $A^{\otimes k} \underline{\otimes} A^{\otimes l}$ . Then on  $A^{\otimes i} \underline{\otimes} A^{\otimes j}$ , we have

$$\begin{aligned} \text{adj}(\Phi)_{(i,j)} &= \text{adj}((\beta_{1,?} \otimes \text{id}_C^{\otimes i+j-1-?})(\text{id}_C \otimes \Phi) + (\text{id}_C \otimes \Phi) \\ &\quad + (\text{id}_C \otimes \beta_{1,?} \otimes \text{id}_C^{\otimes i+j-1-?})(\Delta \otimes \Phi))_{(i,j)} \\ &= (\text{id}_A \otimes \text{adj}(\Phi)_{(i,j-1)})(\alpha_{i,1} \otimes \text{id}_A^{\otimes j-1}) \\ &\quad + (\text{id}_A \otimes \text{adj}(\Phi)_{(i-1,j)}) \\ &\quad + (\text{adj}(\Delta) \otimes \text{adj}(\Phi)_{(i-1,j-1)})(\text{id}_A \otimes \beta_{i-1,1} \otimes \text{id}_A^{\otimes j-1}). \end{aligned}$$



This shows that the map  $\text{adj}(\Phi)$  shares the same inductive formula with the quantum quasi-shuffle product built on  $(A, \text{adj}(\Delta), \tau)$ . Hence we have the conclusion.  $\square$

## 7. REPRESENTATIONS FROM ROTA-BAXTER ALGEBRAS

In order to provide representations of quantum quasi-shuffle algebras, our main point is to construct algebra homomorphisms from quantum quasi-shuffles to some algebras. Inspired by the works on the connection between mixable shuffle algebras and Rota-Baxter algebras ([8] and [6]), we choose the target algebras to be relevant object of Rota-Baxter algebras in braided categories with some suitable conditions. We first recall the definition of Rota-Baxter algebras.

**Definition 7.1.** Let  $\lambda$  be an element in  $\mathbb{K}$ . A pair  $(R, P)$  is called a *Rota-Baxter algebra of weight  $\lambda$*  if  $R$  is an algebra and  $P$  is a linear endomorphism of  $R$  satisfying that for any  $x, y \in R$ ,

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy).$$

Such a map  $P$  is called a *Rota-Baxter operator of weight  $\lambda$* .

We extend this notion in braided categories.

**Definition 7.2.** A triple  $(R, P, \sigma)$  is called a *right (resp. left) weak braided Rota-Baxter algebra of weight  $\lambda$*  if  $(R, \sigma)$  is a braided algebra and  $P$  is a Rota-Baxter operator on  $R$  of weight  $\lambda$  such that  $\sigma(P \otimes \text{id}_R) = (\text{id}_R \otimes P)\sigma$  (resp.  $\sigma(\text{id}_R \otimes P) = (P \otimes \text{id}_R)\sigma$ ). If the multiplication  $m$  of  $R$  satisfies  $m \circ \sigma = m$ ,  $R$  is said to be *commutative*.

Obviously, any usual Rota-Baxter algebra is right weak braided with respect to the usual flip map. Note that a right weak braided Rota-Baxter algebra which is also left weak is a braided Rota-Baxter algebra (cf. [12]). Examples of right or left weak braided Rota-Baxter algebras can be found in [12].

**Theorem 7.3.** Suppose that  $(A, m, \sigma)$  is a braided algebra,  $(R, P, \sigma')$  is a commutative right weak braided Rota-Baxter algebra of weight 1,  $f : A \rightarrow R$  is an algebra map such that  $(f \otimes f) \circ \sigma = \sigma' \circ (f \otimes f)$ . We define a linear map  $\bar{f} : T(A) \rightarrow R$  recursively by  $\bar{f}(1) = 1_R$ ,  $\bar{f}(a_1) = P(f(a_1))$  and  $\bar{f}(a_1 \otimes \cdots \otimes a_n) = P(f(a_1)\bar{f}(a_2 \otimes \cdots \otimes a_n))$  for  $n \geq 2$ . Then  $\bar{f}$  is an algebra map from the quantum quasi-shuffle algebra  $T_\sigma^{qsh}(A)$  to  $R$ .

*Proof.* We denote by  $m'$  the multiplication of  $R$ . For any  $i, j \geq 0$ , we will verify that  $\bar{f} \bowtie_\sigma = m'(\bar{f} \otimes \bar{f})$  on  $A^{\otimes i} \otimes A^{\otimes j}$ . We use induction on  $n = i + j$ .

When  $i = 0$  or  $j = 0$ , it is trivial. So we assume that  $i \neq 0$  and  $j \neq 0$  in the sequel.

For  $n = 2$ , i.e.,  $i = j = 1$ , assume  $a, b \in A$ , then

$$\begin{aligned} \bar{f}(a \bowtie_\sigma b) &= \bar{f}(a \otimes b + \sigma(a \otimes b) + m(a \otimes b)) \\ &= P(f(a)\bar{f}(b)) + Pm'(f \otimes P f)\sigma(a \otimes b) + P(f(ab)) \end{aligned}$$

$$\begin{aligned}
&= P(f(a)P(f(b))) + Pm'(\text{id}_R \otimes P)\sigma'(f \otimes f)(a \otimes b) + P(f(a)f(b)) \\
&= P(f(a)P(f(b))) + Pm'\sigma'(P \otimes \text{id}_R)(f(a) \otimes f(b)) + P(f(a)f(b)) \\
&= P(f(a)P(f(b))) + Pm'(P \otimes \text{id}_R)(f(a) \otimes f(b)) + P(f(a)f(b)) \\
&= P(f(a)P(f(b))) + P(P(f(a))f(b)) + P(f(a)f(b)) \\
&= P(f(a))P(f(b)) \\
&= \bar{f}(a)\bar{f}(b).
\end{aligned}$$

Assume that  $n > 2$  and the result holds for  $i + j < n$ . Note that by an easy induction we have  $(f \otimes \bar{f})\beta_{k,1} = \sigma'(\bar{f} \otimes f)$  for any  $k \in \mathbb{N}$ .

For  $i = n - 2$  and  $x \in A^{\otimes i}$ ,

$$\begin{aligned}
&\bar{f}((a \otimes x) \bowtie_{\sigma} b) \\
&= \bar{f}(a \otimes (x \bowtie_{\sigma} b) + \beta_{i+1,1}(a_1 \otimes x \otimes b) \\
&\quad + (m \otimes \text{id}_A^{\otimes i})(\text{id}_A \otimes \beta_{i,1})(a \otimes x \otimes b)) \\
&= P(f(a)\bar{f}(x \bowtie_{\sigma} b)) \\
&\quad + Pm'(\text{id}_R \otimes Pm')(f \otimes f \otimes \bar{f})(\sigma \otimes \text{id}_A^{\otimes i})(\text{id}_A \otimes \beta_{i,1})(a_1 \otimes x \otimes b) \\
&\quad + Pm'(fm \otimes \bar{f})(\text{id}_A \otimes \beta_{i,1})(a \otimes x \otimes b) \\
&= P(f(a)\bar{f}(x)\bar{f}(b)) \\
&\quad + Pm'(\text{id}_R \otimes Pm')(\sigma' \otimes \text{id}_R)(f \otimes (f \otimes \bar{f})\beta_{i,1})(a_1 \otimes x \otimes b) \\
&\quad + Pm'(m'(f \otimes f) \otimes \bar{f})(\text{id}_A \otimes \beta_{i,1})(a \otimes x \otimes b) \\
&= P(f(a)\bar{f}(x)P(f(b))) \\
&\quad + Pm'(\text{id}_R \otimes Pm')(\sigma' \otimes \text{id}_R)(f \otimes \sigma'(\bar{f} \otimes f))(a_1 \otimes x \otimes b) \\
&\quad + Pm'(m' \otimes \text{id}_R)(f \otimes (f \otimes \bar{f})\beta_{i,1})(a \otimes x \otimes b) \\
&= P(f(a)\bar{f}(x)P(f(b))) + Pm'(\text{id}_R \otimes P)(\text{id}_R \otimes m')\sigma'_1\sigma'_2(f(a_1) \otimes \bar{f}(x) \otimes f(b)) \\
&\quad + Pm'(\text{id}_R \otimes m')(f \otimes \sigma'(\bar{f} \otimes f))(a \otimes x \otimes b) \\
&= P(f(a)\bar{f}(x)P(f(b))) + Pm'(\text{id}_R \otimes P)\sigma'(m' \otimes \text{id}_R)(f(a_1) \otimes \bar{f}(x) \otimes f(b)) \\
&\quad + Pm'(\text{id}_R \otimes m'\sigma')(f \otimes \bar{f} \otimes f)(a \otimes x \otimes b) \\
&= P(f(a)\bar{f}(x)P(f(b))) + P(P(f(a)\bar{f}(x))f(b)) + P(f(a)\bar{f}(y)f(b)) \\
&= P(f(a)\bar{f}(x))P(f(b)) \\
&= \bar{f}(a \otimes x)\bar{f}(b).
\end{aligned}$$

For  $j = n - 2$  and  $y \in A^{\otimes j}$ ,

$$\begin{aligned}
& \overline{f}(a \bowtie_{\sigma} (b \otimes y)) \\
&= \overline{f}(a \otimes b \otimes y + (\text{id}_A \otimes \bowtie_{\sigma(1,j)})(\beta_{1,1} \otimes \text{id}_A^{\otimes j})(a \otimes b \otimes y) + (ab) \otimes y) \\
&= P\left(f(a)P(f(b)\overline{f}(y))\right) + Pm'(f \otimes \overline{f} \bowtie_{\sigma(1,j)})(\sigma \otimes \text{id}_A^{\otimes j})(a \otimes b \otimes y) \\
&\quad + P(f(ab)\overline{f}(y)) \\
&= P\left(f(a)P(f(b)\overline{f}(y))\right) + Pm'(f \otimes m'(\overline{f} \otimes \overline{f}))(\sigma \otimes \text{id}_A^{\otimes j})(a \otimes b \otimes y) \\
&\quad P(f(a)f(b)\overline{f}(y)) \\
&= P\left(f(a)P(f(b)\overline{f}(y))\right) + P(f(a)f(b)\overline{f}(y)) \\
&\quad + Pm'(\text{id}_R \otimes m')(f \otimes Pf \otimes \overline{f})(\sigma \otimes \text{id}_A^{\otimes j})(a \otimes b \otimes y) \\
&= P\left(f(a)P(f(b)\overline{f}(y))\right) + P(f(a)f(b)\overline{f}(y)) \\
&\quad + Pm'(m' \otimes \text{id}_R)((f \otimes Pf)\sigma \otimes \overline{f})(a \otimes b \otimes y) \\
&= P\left(f(a)P(f(b)\overline{f}(y))\right) + P(f(a)f(b)\overline{f}(y)) \\
&\quad + Pm'(m'\sigma' \otimes \text{id}_R)(P \otimes \text{id}_R^{\otimes 2})(f \otimes f \otimes \overline{f})(a \otimes b \otimes y) \\
&= P\left(f(a)P(f(b)\overline{f}(y))\right) + P\left(P(f(a))f(b)\overline{f}(y)\right) + P(f(a)f(b)\overline{f}(y)) \\
&= P(f(a))P(f(b)\overline{f}(y)) \\
&= \overline{f}(a)\overline{f}(b \otimes y).
\end{aligned}$$

Finally, for  $i, j \geq 1$  with  $i + j = n - 2$  and  $x \in A^{\otimes i}, y \in A^{\otimes j}$ ,

$$\begin{aligned}
& \overline{f}((a \otimes x) \bowtie_{\sigma} (b \otimes y)) \\
&= \overline{f}(a \otimes (x \bowtie_{\sigma} (b \otimes y)) + (\text{id}_A \otimes \bowtie_{\sigma(i+1,j)})(\beta_{i+1,1} \otimes \text{id}_A^{\otimes j})(a \otimes x \otimes b \otimes y) \\
&\quad + (m \otimes \bowtie_{\sigma(i,j)})(\text{id}_A \otimes \beta_{i,1} \otimes \text{id}_A^{\otimes j})(a \otimes x \otimes b \otimes y) \\
&= P\left(f(a)\overline{f}(x \bowtie_{\sigma} (b \otimes y))\right) \\
&\quad + Pm'(f \otimes \overline{f} \bowtie_{\sigma(i+1,j)})(\beta_{i+1,1} \otimes \text{id}_A^{\otimes j})(a \otimes x \otimes b \otimes y) \\
&\quad + Pm'(fm \otimes \overline{f} \bowtie_{\sigma(i,j)})(\text{id}_A \otimes \beta_{i,1} \otimes \text{id}_A^{\otimes j})(a \otimes x \otimes b \otimes y) \\
&= P(f(a)\overline{f}(x)\overline{f}(b \otimes y)) \\
&\quad + Pm'(f \otimes m'(\overline{f} \otimes \overline{f}))(\beta_{i+1,1} \otimes \text{id}_A^{\otimes j})(a \otimes x \otimes b \otimes y) \\
&\quad + Pm'(m'(f \otimes f) \otimes m'(\overline{f} \otimes \overline{f}))(\text{id}_A \otimes \beta_{i,1} \otimes \text{id}_A^{\otimes j})(a \otimes x \otimes b \otimes y) \\
&= P\left(f(a)\overline{f}(x)P(f(b)\overline{f}(y))\right) \\
&\quad + Pm'(\text{id}_R \otimes m')((f \otimes \overline{f})\beta_{i+1,1} \otimes \overline{f})(a \otimes x \otimes b \otimes y)
\end{aligned}$$

$$\begin{aligned}
& +Pm'(m' \otimes m')(f \otimes (f \otimes \bar{f})\beta_{i,1} \otimes \bar{f})(a \otimes x \otimes b \otimes y) \\
= & P\left(f(a)\bar{f}(x)P(f(b)\bar{f}(y))\right) \\
& +Pm'(m' \otimes \text{id}_R)(\sigma'(\bar{f} \otimes f) \otimes \bar{f})(a \otimes x \otimes b \otimes y) \\
& +Pm'(m' \otimes m')(f \otimes \sigma'(\bar{f} \otimes f) \otimes \bar{f})(a \otimes x \otimes b \otimes y) \\
= & P\left(f(a)\bar{f}(x)P(f(b)\bar{f}(y))\right) + P(\bar{f}(a \otimes x)f(b)\bar{f}(y)) \\
& +P(f(a)\bar{f}(x)f(b)\bar{f}(y)) \\
= & P\left(f(a)\bar{f}(x)P(f(b)\bar{f}(y))\right) + P\left(P(f(a)\bar{f}(x))f(b)\bar{f}(y)\right) \\
& +P(f(a)\bar{f}(x)f(b)\bar{f}(y)) \\
= & P(f(a)\bar{f}(x))P(f(b)\bar{f}(y)) \\
= & \bar{f}(a \otimes x)\bar{f}(b \otimes y),
\end{aligned}$$

where the sixth equality follows from the fact that  $m'(m' \otimes m')(\text{id}_R \otimes \sigma' \otimes \text{id}_R) = m'(m' \otimes m')$ .  $\square$

*Remark 7.4.* 1. The quantum quasi-shuffle algebra  $T_\sigma^{qsh}(A)$  built on a braided algebra  $(A, m, \sigma)$  has another inductive formula (cf. [14]): for  $i, j > 1$  and any  $a_1, \dots, a_i, b_1, \dots, b_j \in A$ , we have

$$\begin{aligned}
& (a_1 \otimes \dots \otimes a_i) \bowtie_\sigma (b_1 \otimes \dots \otimes b_j) \\
= & \left( (a_1 \otimes \dots \otimes a_i) \bowtie_\sigma (b_1 \otimes \dots \otimes b_{j-1}) \right) \otimes b_j \\
& + (\bowtie_{\sigma(i-1,j)} \otimes \text{id}_A)(\text{id}_A^{\otimes i-1} \otimes \beta_{1,j})(a_1 \otimes \dots \otimes a_i \otimes b_1 \otimes \dots \otimes b_j) \\
& + (\bowtie_{\sigma(i-1,j-1)} \otimes m)(\text{id}_A^{\otimes i-1} \otimes \beta_{1,j-1} \otimes \text{id}_A)(a_1 \otimes \dots \otimes a_i \otimes b_1 \otimes \dots \otimes b_j).
\end{aligned}$$

If  $(R, P, m', \sigma')$  is a commutative left weak braided Rota-Baxter algebra of weight 1,  $f : A \rightarrow R$  is an algebra map such that  $(f \otimes f)\sigma = \sigma'(f \otimes f)$ , then the linear map  $\bar{f} : T(V) \rightarrow R$  defined recursively by  $\bar{f} = Pm'(\bar{f} \otimes f)$  is also an algebra map from the quantum quasi-shuffle algebra  $T_\sigma^{qsh}(A)$  to  $R$ . The proof is similar to the one of the above theorem.

2. By composing the left regular representation of  $R$ , Theorem 7.3 provides a representation of  $T_\sigma^{qsh}(A)$  on  $R$ .

3. Restricting the map  $\bar{f}$  to the subalgebra  $S_\sigma^{qsh}(A)$  of  $T_\sigma^{qsh}(A)$ , we obtain an algebra map from  $S_\sigma^{qsh}(A)$  to  $R$ , and hence a representation of  $S_\sigma^{qsh}(A)$  on  $R$ .

**Example 7.5.** Let  $(A, m, \sigma)$  be a braided algebra such that  $m\sigma = m$ . We define  $P : A \rightarrow A$  by  $P(a) = -a$ . One can verify that  $(A, m, P, \sigma)$  is a commutative right weak braided Rota-Baxter algebra of weight 1. If we define  $f$  to be the identity map on  $A$ , then the map  $\bar{f} : T_\sigma^{qsh}(A) \rightarrow A$  given by  $\bar{f}(a_1 \otimes \dots \otimes a_n) = (-1)^n a_1 \dots a_n$  is an algebra map. So  $T_\sigma^{qsh}(A)$  acts on  $A$  by  $(a_1 \otimes \dots \otimes a_n) \cdot a = (-1)^n a_1 \dots a_n a$ . In addition, if  $A$  is unital, then it follows immediately that  $A$  is a cyclic  $T_\sigma^{qsh}(A)$ -module generated by  $1_A$ .

**Example 7.6.** Let  $q \in \mathbb{K}$  be an invertible number which is not a root of unity. Consider the polynomial algebra  $\mathbb{K}[t]$  as a commutative braided algebra with the flip  $\sigma(t^n \otimes t^m) = t^m \otimes t^n$ . Then the quantum quasi-shuffle algebra built on  $\mathbb{K}[t]$  is just the usual quasi-shuffle algebra  $T^{qsh}(\mathbb{K}[t])$  of  $\mathbb{K}[t]$ . We define  $P : \mathbb{K}[t] \rightarrow \mathbb{K}[t]$  by  $P(t^n) = \frac{q^n t^n}{1-q^n}$ . Then  $P$  is a Rota-Baxter operator of weight 1 on  $\mathbb{K}[t]$  (cf. [7]). Moreover,  $(\mathbb{K}[t], P, \sigma)$  is a commutative right weak braided Rota-Baxter algebra of weight 1. If we define  $f$  to be the identity map on  $\mathbb{K}[t]$ , then the map  $\bar{f} : T^{qsh}(\mathbb{K}[t]) \rightarrow \mathbb{K}[t]$  given by

$$\bar{f}(t^{i_1} \otimes \dots \otimes t^{i_n}) = \frac{q^{ni_n + (n-1)i_{n-1} + \dots + i_1} t^{i_1 + \dots + i_n}}{(1 - q^{i_n})(1 - q^{i_n + i_{n-1}}) \dots (1 - q^{i_n + \dots + i_1})}$$

is an algebra map. So  $T^{qsh}(\mathbb{K}[t])$  acts on  $\mathbb{K}[t]$  by

$$(t^{i_1} \otimes \dots \otimes t^{i_n}) \cdot t^m = \frac{q^{ni_n + (n-1)i_{n-1} + \dots + i_1} t^{i_1 + \dots + i_n + m}}{(1 - q^{i_n})(1 - q^{i_n + i_{n-1}}) \dots (1 - q^{i_n + \dots + i_1})}.$$

From the above formula, we see that  $\mathbb{K}[t]$  is a cyclic  $T^{qsh}(\mathbb{K}[t])$ -module generated by 1. As a consequence, for any  $k$ ,  $\mathbb{K}[t]t^k$  is again a  $T^{qsh}(\mathbb{K}[t])$ -module.

Finally, by using Example 7.8, we reformulate the expression of multiple  $q$ -zeta values. We will replace every term in the multiple  $q$ -zeta values by an image of the algebra map  $\bar{f}$  on some element of the subalgebra of quasi-shuffle algebra  $T^{qsh}(\mathbb{K}[t])$  generated by  $\mathbb{K}[t]$ .

From now on, we assume  $q \in \mathbb{K}$  is an invertible number which is not a root of unity.

Let  $(\mathbb{K}[t], P)$  be the commutative Rota-Baxter algebra given in Example 7.6. Consider the subalgebra  $S^{qsh}(\mathbb{K}[t])$  of  $T^{qsh}(\mathbb{K}[t])$  generated by  $\mathbb{K}[t]$ . We denote by  $*$  the usual quasi-shuffle product on  $T^{qsh}(\mathbb{K}[t])$ . Note that for any  $n, i \in \mathbb{N}$ , we have

$$\bar{f}((t^n)^{*i}) = \bar{f}(t^n)^i = \frac{q^{ni} t^{ni}}{(1 - q^n)^i}.$$

Given  $k \in \mathbb{N}$  and any multi-index  $(i_1, \dots, i_k) \in \mathbb{N}^k$ , we set

$$Z_q(i_1, \dots, i_k; t) = \sum_{n_1 > \dots > n_k > 0} \bar{f}((t^{n_1})^{*i_1} * \dots * (t^{n_k})^{*i_k}),$$

where the sum is over all multi-index  $(n_1, \dots, n_k) \in \mathbb{N}^k$  with the indicated inequalities. Then after evaluating at 1, we have

$$Z_q(i_1, \dots, i_k) = Z_q(i_1, \dots, i_k; 1) = \sum_{n_1 > \dots > n_k > 0} \frac{q^{i_1 n_1} \dots q^{i_k n_k}}{(1 - q^{n_1})^{i_1} \dots (1 - q^{n_k})^{i_k}}.$$

The series  $Z_q(i_1, \dots, i_k)$  is Zudilin's  $q$ -analogue of multiple zeta values (cf. [29]).

Now we turn to Bradley's multiple  $q$ -zeta values ([2]). Given  $k \in \mathbb{N}$ , a multi-index  $(i_1, \dots, i_k) \in \mathbb{N}^k$  is called *admissible* if  $i_1 \geq 2$ . For any admissible index  $(i_1, \dots, i_k)$ , the *multiple  $q$ -zeta value*  $\zeta(i_1, \dots, i_k)$  is defined as follows:

$$\zeta_q(i_1, \dots, i_k) = \sum_{n_1 > \dots > n_k \geq 1} \frac{q^{(i_1-1)n_1} \dots q^{(i_k-1)n_k}}{[n_1]_q^{i_1} \dots [n_k]_q^{i_k}}.$$

By using the algebra map  $\overline{f}$ , we define a formal series  $\zeta_q(i_1, \dots, i_k; t) \in \mathbb{K}[[t]]$  as follows:

$$\zeta_q(i_1, \dots, i_k; t) = \sum_{n_1 > \dots > n_k \geq 1} \frac{\overline{f}\left(\left((1-q)t^{n_1}\right)^{*i_1} * \dots * \left((1-q)t^{n_k}\right)^{*i_k}\right)}{q^{n_1 + \dots + n_k}},$$

where the sum is over all multi-index  $(n_1, \dots, n_k) \in \mathbb{N}^k$  with the indicated inequalities. By evaluating at 1, we have that  $\zeta_q(i_1, \dots, i_k; 1) = \zeta_q(i_1, \dots, i_k)$ .

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